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Relationship between Symmetries and Conservation Laws

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The fundamental relation between Lie–Bäcklund symmetry generators and conservation laws of an arbitrary differential equation is derived without regard to a Lagrangian formulation of the differential equation. This relation is used in the construction of conservation laws for partial differential equations irrespective of the knowledge or existence of a Lagrangian. The relation enables one to associate symmetries to a given conservation law of a differential equation. Applications of these results are illustrated for a range of examples.

1. INTRODUCTION

In her paper on symmetries and conservation laws, Emmy Noether (1918) proved that, for Euler–Lagrange differential equations, to each Noether symmetry associated with a Lagrangian there corresponds a conservation law which can be determined explicitly by means of a formula. The relationship between the components of the Noether conserved vector T and the Lie–Bäcklund operator which generates a Noether symmetry that gives rise to T, *inter alia*, was investigated in Ibragimov *et al.* (1998). The question then naturally arises of whether a similar result applies to differential equations that do not admit of a Lagrangian formulation. Indeed, it is well known that there exist differential equations which are not derivable from a variational principle, e.g., evolution-type equations (see, e.g., Anderson and Duchamp, 1984).

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In a recent paper on symmetries (local and nonlocal) and conservation laws, Anco and Bluman (1996) derived an identity which does not depend on the use of a Lagrangian and provides a correspondence between symmetries and conservation laws for self-adjoint differential equations.

In this paper, we present the relationship between the components of a conserved vector of an *arbitrary* differential equation and the Lie–Bäcklund symmetry generator of the equation which is *associated* with the conserved vector components. Some aspects of this result were first reported at the Modern Group Analysis Conference in 1997 (Kara and Mahomed, 1999). For differential equations with a small parameter, we have proved analogous results and presented applications in Kara *et al.* (1999). A result that relates symmetries and conservation laws has significant applications. First, in the absence of a Lagrangian, the construction of a conservation law of a given equation is generally attempted by means of the *direct method*. This involves the expansion of the conservation law equation, subject to the differential equation being satisfied, as a determining equation for the components of the conserved vector. This approach was first used by Laplace (1798) (see discussion in Anderson and Ibragimov, 1994, Chapter 6) in the derivation of the components of the well-known Laplace or Laplace–Runge–Lenz vector of the classical two-body Kepler problem. Our result presented here imposes a natural symmetry condition which together with the direct method simplifies the solution procedure for the determination of a conservation law. The method given here provides a direct link between symmetries and conservation laws. Second, the relationship between symmetry and conservation law given here enables one to obtain the symmetry (or symmetries) associated with a given conservation law. In the case of ordinary differential equations, this provides for a double reduction of order (Kara *et al.*, 1994). Third, special cases of the result given here relating to a variational formulation can be used for the construction of Lagrangians of partial differential equations (Ibragimov *et al.*, 1998).

Apart from the applications alluded to above, further examples are given in Section 4 which amply illustrate our method.

We briefly outline the notation and pertinent results used in this work. In this regard, the reader is referred to Anderson and Duchamp (1984). The convention that repeated indices imply summation is used.

Let $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ be the independent variable with coordinates x^i, and let $u = (u^1, u^2, \ldots, u^m) \in \mathbb{R}^m$ be the dependent variable with coordinates u^α. Furthermore, let π: $\mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection map $\pi(x, u) = x$. Also, suppose that s: $\chi \subset \mathbb{R}^n \to \mathcal{U} \subset \mathbb{R}^{n+m}$ is a smooth map such that $\pi \circ s = 1_\chi$, where 1_χ is the identity map on χ. The r-jet bundle $J^r(\mathcal{U})$ is given by the equivalence classes of sections of $\mathcal{U}$. The coordinates on $J^r(\mathcal{U})$ are denoted by $(x^i, u^\alpha, \ldots, u^\alpha_{i_1 \ldots i_r})$, where $1 \leq i_1 \leq \ldots \leq i_r \leq n$

and $u^{\alpha}_{i_1 \ldots i_r}$ corresponds to the partial derivatives of u^{α} with respect to $x^{i_1}, \ldots, x^{i_r}$. The partial derivatives of u with respect to x are connected by the operator of total differentiation

$$D_i = \frac{\partial}{\partial x^i} + u^{\alpha}_i \frac{\partial}{\partial u^{\alpha}} + u^{\alpha}_{ij} \frac{\partial}{\partial u^{\alpha}_j} + \cdots, \qquad i = 1, \ldots, n$$

as

$$u^{\alpha}_i = D_i(u^{\alpha}), \qquad u^{\alpha}_{ij} = D_j D_i(u^{\alpha}), \ldots$$

The collection of all first-order derivatives u^{α}_i will be denoted by $u_{(1)}$. Similarly, the collections of all higher order derivatives will be denoted by $u_{(2)}$, $u_{(3)}$, ... The r-jet bundle on $\mathcal{U}$ will be written as $J^r(\mathcal{U}) = \{(x, u, u_{(1)}, \ldots, u_{(r)})/(x, u) \in \mathcal{U}\}$.

We now review the space of differential forms on $J^r(\mathcal{U})$. To this end, let $\Omega^r_k(\mathcal{U})$ be the vector space of *differential k-forms* on $J^r(\mathcal{U})$ with *differential* d. A smooth differential k-form on $J^r(\mathcal{U})$ is given by

$$\omega = f_{i_1 i_2 \ldots i_k}\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

where each component $f_{i_1 i_2 \ldots i_k} \in \Omega^r_0(\mathcal{U})$, i.e., $f_{i_1 i_2 \ldots i_k} = f_{i_1 i_2 \ldots i_k}(x, u, u_{(1)}, \ldots, u_{(r)})$. Note that for differential functions $f \in \Omega^r_0(\mathcal{U})$,

$$Df = D_j f\, dx^j \tag{1.1}$$

where D is the *total differential* or the *total exterior derivative*. Moreover, the total exterior derivative of ω is $D\omega = Df_{i_1 i_2 \ldots i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$ and by invoking (1.1) one has

$$D\omega = D_j f_{i_1 i_2 \ldots i_k}\, dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

The total differential D has properties analogous to the algebraic properties of the usual exterior derivative d:

$$D(\omega \wedge \nu) = D\omega \wedge \nu + (-1)^k \omega \wedge D\nu$$

for ω a k-form and ν an l-form and $D(D\omega) = 0$. Also, it is known that if $D\omega = 0$, then ω is a locally exact k-form, i.e., $\omega = D\nu$ for some $(k-1)$-form ν (Anderson and Duchamp, 1980).

2. ACTION OF SYMMETRIES

Consider an rth-order system of partial differential equations of n independent and m dependent variables,

$$E^{\beta}(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \qquad \beta = 1, \ldots, \tilde{m} \tag{2.1}$$

Definition 1. A *conserved form* of (2.1) is a differential $(n - 1)$-form

$$\omega = T^i(x, u, u_{(1)}, \ldots, u_{(r-1)})\left(\frac{\partial}{\partial x^i} \rfloor (\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)\right) \tag{2.2}$$

defined on $J^{r-1}(\mathcal{U})$ if

$$D\omega = 0 \tag{2.3}$$

is satisfied on the surface given by (2.1).

Remark. When Definition 1 is satisfied, (2.3) is called a *conservation law* for (2.1).

It is clear that (2.3) evaluated on the surface (2.1) implies

$$D_i T^i = 0 \tag{2.4}$$

on the surface given by (2.1), which is also referred to as a conservation law of (2.1). The tuple $T = (T^1, \ldots, T^n)$, $T^j \in \Omega_0^{r-1}(\mathcal{U})$, $j = 1, \ldots, n$, is called a *conserved vector* of (3.1).

We now review some definitions and results relating to Euler–Lagrange, Lie–Bäcklund, and Noether operators (Ibragimov, 1985; Ibragimov *et al.*, 1998, and references therein).

Let $\mathcal{A} = \cup_{r=0}^{p} \Omega_0^r$ for some $p < \infty$. Then $\mathcal{A}$ is the universal space of differential functions of finite orders.

Consider a Lie–Bäcklund operator given by the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots \tag{2.5}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ and the additional coefficients are determined uniquely by the prolongation formulas

$$\zeta_{i_1 \cdots i_s}^\alpha = D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \cdots i_s}^\alpha \qquad s = 1, 2, \ldots \tag{2.6}$$

In (2.6), W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha \tag{2.7}$$

Let us also mention that if X is a Lie–Bäcklund operator, ω a k-form, and ν an l-form, then

$$X(\omega \wedge \nu) = X(\omega) \wedge \nu + \omega \wedge X(\nu)$$

Lie–Bäcklund operators $\tilde{X}$ and X are said to be equivalent if $X - \tilde{X} = \lambda^i D_i$, $\lambda^i \in \mathcal{A}$. If $\lambda^i = \xi^i$, then $\tilde{X}$ is called a *canonical operator.*

A Lie–Bäcklund operator X is said to be a *Noether symmetry* generator associated with a Lagrangian $L \in \mathcal{A}$ if there exists a vector $B = (B^1, \ldots, B^n)$, $B^i \in \mathcal{A}$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i) \tag{2.8}$$

If in (2.8) $B^i = 0$, $i = 1, \ldots, n$, then X is referred to as a *strict Noether symmetry* generator associated with a Lagrangian $L \in \mathcal{A}$.

In view of the above discussions and definitions, the Noether theorem (Noether, 1918) is formulated as follows.

The Noether Theorem (Noether, 1918). For any Noether symmetry generator X associated with a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector $T = (T^1, \ldots, T^n)$, $T^i \in \mathcal{A}$, defined by

$$T^i = N^i(L) - B^i, \qquad i = 1, \ldots, n \tag{2.9}$$

which is a conserved vector of the Euler–Lagrange equations $\delta L/\delta u^\alpha = 0$, $\alpha = 1, \ldots, m$, where $\delta/\delta u^\alpha$ is the Euler–Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \qquad \alpha = 1, \ldots, m \tag{2.10}$$

and the Noether operator associated with X is

$$N^i = \xi^i + W^\alpha \frac{\partial}{\partial u^\alpha_i} + \sum_{s\geq 1} D_{i_1} \cdots D_{is}(W^\alpha) \frac{\partial}{\partial u^\alpha_{ii_1\cdots i_s}}, \qquad i = 1, \ldots, n$$

in which the Euler–Lagrange operators with respect to derivatives of u^α are obtained from (2.10) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s\geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}},$$

$$i = 1, \ldots, n, \quad \alpha = 1, \ldots, m$$

3. INVARIANCE OF CONSERVED FORMS

We recall (Ibragimov *et al.*, 1998) an important result relating a Noether symmetry X and its corresponding conserved vector $T = (T^1, \ldots, T^n)$, where, the T^i satisfy (2.9).

Theorem 1 (Ibragimov *et al.*, 1998). The components of the Noether conserved vector T, given by (2.9), associated with the Lie–Bäcklund operator X, which is a generator of a Noether symmetry, satisfy

$$X(T^i) + D_k(\xi^k)T^i - T^k D_k(\xi^i) = N^i(D_k(B^k)) + B^k D_k(\xi^i) - D_k(\xi^k)B^i - X(B^i),$$
$$i = 1, \ldots, n \tag{3.1}$$

In Ibragimov *et al.* (1998) it was proved, *inter alia*, that *any Noether symmetry is equivalent to a strict Noether symmetry*. Precisely, one can state the following theorem:

Theorem 2 (Ibragimov *et al.*, 1998). If a Lie–Bäcklund operator X satisfies (2.8), then its equivalent operator

$$\tilde{X} = X - \frac{1}{L} B^i D_i = \left(\xi^i - \frac{1}{L} B^i\right) \frac{\partial}{\partial x^i} + \left(\eta^\alpha - \frac{1}{L} B^i u_i^\alpha\right) \frac{\partial}{\partial u^\alpha} + \cdots$$

satisfies

$$\tilde{X}L + LD_i\tilde{\xi}^i = 0$$

where $\tilde{\xi}^i = \xi^i - (1/L)B^i$ for $i = 1, \ldots, n$.

Consequently, (3.1) in Theorem 1 can be written independently of B^i and N^i, i.e., we have the result

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \qquad i = 1, \ldots, n \tag{3.2}$$

The relation (3.2) connects a Noether symmetry generator to components of a conserved vector given by (2.9). It should be borne in mind that (3.2) is deduced from a Lagrangian formulation. The question is whether the result (3.2) holds without regard to the knowledge or existence of a Lagrangian of a given differential equation. The answer is provided by the main theorem proved below.

We first state the following definition, which gives meaning to the invariance of a differential form under a Lie–Bäcklund group (see Anderson and Ibragimov, 1995, 1996, for accounts on these groups).

Definition 2. The differential k-form $\omega = f_{i_1 i_2 \ldots i_k}(x, u, u_{(1)}, \ldots, u_{(r)})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ is called an *invariant form of order k* with respect to the Lie–Bäcklund transformation group $\bar{x}^i = \exp(\epsilon X)(x^i)$, $\bar{u}^\alpha = \exp(\epsilon X)(u^\alpha)$, $\bar{u}_i^\alpha = \exp(\epsilon X)(u_i^\alpha), \ldots$, $i = 1, \ldots, n$, $\alpha = 1, \ldots, m$, on $\mathcal{A}$, where ϵ is the group parameter, $\exp(\epsilon X) = 1 + \epsilon X + (\epsilon^2/2!)X^2 + (\epsilon^3/3!)X^3 + \cdots$, and X is the Lie–Bäcklund operator (2.5), if

$$\begin{aligned} &f_{i_1 i_2 \cdots i_k}(\bar{x}, \bar{u}, \bar{u}_{(1)}, \ldots, \bar{u}_{(r)}) d\bar{x}^{i_1} \wedge \ldots \wedge d\bar{x}^{i_k} \\ &\quad = f_{i_1 i_2 \cdots i_k}(x, u, u_{(1)}, \ldots, u_{(r)})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \end{aligned}$$

The following lemma is also used in the main theorem.

Lemma. The differential form $\omega = f_{i_1 i_2 \cdots i_k}(x, u, u_{(1)}, \ldots, u_{(r)})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ is an invariant form of the Lie–Bäcklund group with generator X if and only if

$$X(\omega) = 0$$

Proof. We utilize Definition 2. Also we write $\overline{\omega}$ to indicate that ω is in terms of the transformed quantities. Thus, by using the infinitesimal form of the Lie–Bäcklund group and the Taylor expansion, it is not difficult to deduce that

$$\begin{aligned}\overline{\omega} &= f_{i_1 i_2 \cdots i_k}(\overline{x}, \overline{u}, \overline{u}_{(1)}, \ldots, \overline{u}_{(r)})\, d\overline{x}^{i_1} \wedge \ldots \wedge d\overline{x}^{i_k} \\ &= [f_{i_1 i_2 \ldots i_k} + \epsilon X(f_{i_1 i_2 \ldots i_k})]\, d(x^{i_1} + \epsilon \xi^{i_1}) \wedge \ldots \wedge d(x^{i_k} + \epsilon \xi^{i_k}) + O(\epsilon^2) \\ &= \omega + \epsilon X(\omega) + O(\epsilon^2)\end{aligned}$$

Hence if $\overline{\omega} = \omega$, then $X(\omega) = 0$ for ϵ small. For the converse, suppose that $X(\omega) = 0$. Then $\overline{X}(\overline{\omega}) = 0$. Now $(d/d\epsilon)\overline{\omega} = \overline{X}(\overline{\omega})$ is zero. Thus $\overline{\omega}$ is constant, which in turn, using the identity property of the group, implies that $\overline{\omega} = \omega$.

Main Theorem. Suppose that X is a Lie–Bäcklund operator (2.5) such that the form ω given by (2.2) is invariant under X. Then

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \qquad i = 1, \ldots, n \tag{3.3}$$

Proof. Let us now write the $(n - 1)$-form (2.2) as

$$\omega = T^{i_k} \frac{\partial}{\partial x^{i_k}} \rfloor (dx^{i_1} \wedge \ldots \wedge dx^{i_n}) \tag{3.4}$$

Therefore,

$$\begin{aligned}X(\omega) &= X(T^{i_k}) \frac{\partial}{\partial x^{i_k}} \rfloor (dx^{i_1} \wedge \ldots \wedge dx^{i_n}) \\ &\quad + T^{i_k}(-1)^{k-1} X\, (dx^{i_1} \wedge \ldots \widehat{\wedge\, dx^{i_k} \wedge} \ldots \wedge dx^{i_n}) \\ &= X(T^{i_k}) \frac{\partial}{\partial x^{i_k}} \rfloor (dx^{i_1} \wedge \ldots \wedge dx^{i_n}) \\ &\quad + T^{i_k}(-1)^{k-1} \sum_{l=1, l\neq k}^{n} (dx^{i_1} \wedge \ldots \widehat{\wedge\, dx^{i_k} \wedge} \ldots \wedge dx^{i_{l-1}} \\ &\quad \wedge D\xi^{i_l} \wedge dx^{i_{l+1}} \wedge \ldots \wedge dx^{i_n})\end{aligned}$$

where the caret denotes omission. Since for fixed l, $D\xi^{i_l} = D_{i_k}\xi^{i_l}\, dx^{i_k}$ by (1.1), we have

$$\begin{aligned}X(\omega) &= X(T^{i_k}) \frac{\partial}{\partial x^{i_k}} \rfloor (dx^{i_1} \wedge \ldots \wedge dx^{i_n}) \\ &\quad + T^{i_k}(-1)^{k-1} \Bigg\{ \sum_{l=1, l\neq i_k}^{n} D_{i_l}\xi^{i_l} (dx^{i_1} \wedge \ldots \widehat{\wedge\, dx^{i_k} \wedge}\end{aligned}$$

$$\ldots \wedge dx^{i_l-1} \wedge dx^{i_l} \wedge dx^{i_l+1} \wedge \ldots \wedge dx^{i_n})$$

$$+ D_{i_k}\xi^{i_l}(dx^{i_1} \wedge \ldots \widehat{\wedge\, dx^{i_k} \wedge} \ldots \wedge dx^{i_l-1} \wedge dx^{i_k} \wedge dx^{i_l+1} \wedge \ldots \wedge dx^{i_n})\}$$

$$= X(T^{i_k}) \frac{\partial}{\partial x^{i_k}} \rfloor (dx^{i_1} \wedge \ldots \wedge dx^{i_n}) + T^{i_k} D_{i_l}\xi^{i_l} \frac{\partial}{\partial x^{i_k}} \rfloor (dx^{i_1} \wedge \ldots \wedge dx^{i_n})$$

$$- T^{i_l} D_{i_l}\xi^{i_k} \frac{\partial}{\partial x^{i_k}} \rfloor (dx^{i_1} \wedge \ldots \wedge dx^{i_n})$$

Thus, $X(\omega) = 0$ gives

$$X(T^{i_k}) + T^{i_k}D_{i_l}(\xi^{i_l}) - T^{i_l}D_{i_l}(\xi^{i_k}) = 0, \qquad k = 1, \ldots, n$$

which is (3.3).

The implications of this result on differential equations are important and given in the following definition.

Definition 3. A Lie–Bäcklund symmetry generator X is said to be *associated* with a conserved vector T (or its corresponding conserved form ω) of the system (3.1) if X and T satisfy the relations (4.3) [or equivalently if $X(\omega) = 0$].

Corollary 1. Suppose that X is a canonical Lie–Bäcklund symmetry generator of the system (2.1) such that the conserved form ω of (2.1), given by (2.2), is invariant under X. Then

$$X(T^i) = 0, \qquad i = 1, \ldots, n \tag{3.5}$$

In the following corollary, we discuss the above result for a system of ordinary differential equations (one independent variable x).

Corollary 2. Suppose that X is a Lie–Bäcklund symmetry generator of a system of ordinary differential equations and X is associated with a conserved quantity T. Then X is a symmetry generator of T, viz.

$$X(T) = 0 \tag{3.6}$$

It is important to point out that the result $X(T) = 0$ in (3.6) has effectively been applied before in the case of point symmetries to physically interesting systems. The reader is referred to, e.g., Leach (1981).

4. APPLICATIONS

In this section we derive conservation laws for some well-known partial differential equations of mathematical physics using our symmetry conditions (3.3) together with the determining equation for the conservation law (2.4).

In the first example, which admits of a Lagrangian formulation, we compare the results obtained by our method with those obtained by Noether's theorem. The second example deals with Burger's equation. We utilize the direct method to construct conservation laws, as there is no Lagrangian for this equation. The direct method does not provide a correspondence between symmetries and conservation laws. Hence we invoke our method to provide this link. In Example 3 we utilize our method, viz. we invoke the dual conditions (3.3) and (2.4) to find conservation laws which are associated with a single symmetry for each example. In Example 5 we look at the alternate problem of knowing a conservation law and finding the symmetries associated with it.

Example 1. The equation

$$u_{xt} + u_x + u^2 = 0 \tag{4.1}$$

which arises in the study of Maxwellian tails, admits a four-dimensional Lie algebra of point symmetry generators (Euler *et al.*, 1988). One of them (in extended form) is

$$X = \mathrm{e}^t \frac{\partial}{\partial t} - u\mathrm{e}^t \frac{\partial}{\partial u} - (u\mathrm{e}^t + 2u_t\mathrm{e}^t) \frac{\partial}{\partial u_t} - u_x\mathrm{e}^t \frac{\partial}{\partial u_x}$$

We construct a conserved vector $T = (T^1, T^2)$ which has associated with it the symmetry generator X given above. This is done by invoking our result (3.3) together with the condition for the conservation law (2.4) for (4.1). The determining equation for the components and T^1 and T^2, using the direct approach, is

$$(D_tT^1 + D_xT^2)\big|_{(4.1)} = 0$$

Splitting with respect to u_{tt} and u_{xx} gives that T^1 and T^2 are independent of u_t and u_x, respectively. The remaining part of the determining equation becomes (we have replaced u_{xt} by $-u_x - u^2$)

$$\frac{\partial T^1}{\partial t} + \frac{\partial T^1}{\partial u} u_t + \frac{\partial T^1}{\partial u_x} (-u^2 - u_x) + \frac{\partial T^2}{\partial x} + \frac{\partial T^2}{\partial u} u_x + \frac{\partial T^2}{\partial u_t} (-u^2 - u_x) = 0 \tag{4.2}$$

We now impose the symmetry conditions (3.3) with the above X, which are

$$X(T^1) = 0, \qquad X(T^2) + \mathrm{e}^t T^2 = 0$$

This yields

$$\frac{\partial T^1}{\partial t} - \frac{\partial T^1}{\partial u} u - \frac{\partial T^1}{\partial u_x} u_x = 0, \qquad \frac{\partial T^2}{\partial t} - \frac{\partial T^2}{\partial u} u - (2u_t + u) \frac{\partial T^2}{\partial u_t} + T^2 = 0 \tag{4.3}$$

The solution of equation (4.3) gives

$$T^1 = f^1(a, \alpha, \beta), \qquad T^2 = uf^2(a, \alpha, \gamma) \tag{4.4}$$

where the f^i are functions of $a = x$, $\alpha = ue^t$, $\beta = u_x/u$, and $\gamma = u_t e^{2t} + ue^{2t}$. The substitution of (4.4) into (4.2) results in the linear equation

$$\alpha\gamma \frac{\partial f^1}{\partial \alpha} - (\alpha^2 + \beta\gamma) \frac{\partial f^1}{\partial \beta} + \alpha^2 \beta f^2 + \alpha^2 \frac{\partial f^2}{\partial a} + \alpha^3 \beta \frac{\partial f^2}{\partial \alpha} - \alpha^4 \frac{\partial f^2}{\partial \gamma} = 0$$

Since we have the functions $f^1(a, \alpha, \beta)$ and $f^2(a, \alpha, \gamma)$, we can easily obtain a solution of the above equation by differentiation twice with respect to γ and separation with respect to β. This gives

$$\alpha^2 f^2 + \alpha^3 \frac{\partial f^2}{\partial \alpha} = A_1\gamma + B_1, \qquad \alpha^2 \frac{\partial f^2}{\partial a} - \alpha^4 \frac{\partial f^2}{\partial \gamma} = A_2\gamma + B_2$$

where the A_i and B_i are functions of a and α. The substitution of these back into the linear equation and a split with respect to γ results in

$$\alpha \frac{\partial f^1}{\partial \alpha} - \beta \frac{\partial f^1}{\partial \beta} = -A_1\beta - A_2, \qquad -\alpha^2 \frac{\partial f^1}{\partial \beta} = -B_1\beta - B_2$$

The solution of the above pair of equations yields

$$f^1 = \frac{1}{2} \Gamma_1 \alpha^2 \beta^2 + \frac{\beta}{\alpha^2} B_2 + C, \qquad f^2 = -\frac{\gamma}{\alpha^4} B_2 + \frac{1}{3}\Gamma_1 \alpha^2 - \frac{1}{2} \Gamma_2 \frac{\gamma^2}{\alpha} + \frac{\Gamma_3}{\alpha}$$

where

$$-\alpha \frac{\partial C}{\partial \alpha} + \frac{1}{\alpha^2} \frac{\partial B_2}{\partial a} = \Gamma_2 \alpha^3$$

and the Γ_i are constants. One can easily see that a solution is (set $B_2 = \Gamma_1 = \Gamma_3 = 0$, $C = \alpha^3/3$, and $\Gamma_2 = -1$) $f^1 = 1/3\alpha^3$, $f^2 = \gamma^2/(2\alpha)$. Then T^1 and T^2 are

$$T^1 = \frac{1}{3} u^3 e^{3t}, \qquad T^2 = \frac{1}{2} e^{3t}(u + u_t)^2$$

which constitute a nontrivial conservation law associated with X given above.

Another conservation law (not equivalent to the first one) associated with X corresponds to (set $\Gamma_1 = 1$ and the rest of the terms zero)

$$T^1 = \frac{1}{2}\, e^{2t} u_x^2, \qquad T^2 = \frac{1}{3}\, e^{2t} u^3$$

We see here that there could arise more than one conservation law associated with a given symmetry.

Consider next the scaling symmetry generator

$$X = x\,\frac{\partial}{\partial x} - u\,\frac{\partial}{\partial u} - u_t\,\frac{\partial}{\partial u_t} - 2u_x\,\frac{\partial}{\partial u_x}$$

which is admitted by (4.1) (Euler *et al.*, 1988). The symmetry conditions (3.3) on T^1 and T^2 are

$$X(T^1) + T^1 = 0, \qquad X(T^2) = 0$$

The resulting solutions of these give

$$T^1 x = f^1(a, \alpha, \beta, \gamma), \qquad T^2 = f^2(a, \alpha, \beta, \gamma)$$

where $a = t$, $\alpha = xu$, $\beta = xu_t$, and $\gamma = x^2\, u_x$. Since T^1 and T^2 are independent of u_t and u_x, respectively (this is a consequence of the determining equation for the conserved components given above), T^1 is independent of β and T^2 of γ. The condition (4.2) then implies the linear equation

$$\frac{\partial f^1}{\partial a} + \beta\,\frac{\partial f^1}{\partial \alpha} - (\alpha^2 + \gamma)\,\frac{\partial f^1}{\partial \gamma} + \alpha\,\frac{\partial f^2}{\partial \alpha} + \beta\,\frac{\partial f^2}{\partial \beta} + \gamma\,\frac{\partial f^2}{\partial \alpha} - (\alpha^2 + \gamma)\,\frac{\partial f^2}{\partial \beta} = 0$$

By differentiation twice with respect to β and thereafter a split with respect to γ we obtain

$$\frac{\partial f^2}{\partial \alpha} - \frac{\partial f^2}{\partial \beta} = A_1\beta + B_1, \qquad \alpha\,\frac{\partial f^2}{\partial \alpha} + (\beta - \alpha^2)\,\frac{\partial f^2}{\partial \beta} = A_2\beta + B_2$$

where the A_i and B_i are functions of a and α. The insertion of the last pair of equations into the linear equation gives the equations for f^1. The solution of the equations in f^1 and f^2 gives trivial conservation law of (4.1).

The translation in x symmetry generated by $X = \partial/\partial x$ is admitted by (4.1) and it is straightforward to obtain the conserved components associated with X. The calculations are routine as before and we obtain

$$T^1 = \frac{2}{3}\, A_1 u^3\, e^{3t} + 2A_1\, e^{3t}\, uu_x + \frac{1}{2}\, A_2\, e^{2t}\, u_x^2 + B\,\frac{u_x}{u^2},$$

$$T^2 = A_1\, e^{3t}\, u_t^2 - B\,\frac{u_t}{u^2} + C$$

where the A_i are constants and $B(t,u)$ and $C(t,u)$ satisfy $\partial B/\partial t + u^2\partial C/\partial u = u^4A_2e^{2t} - 4u^3A_1e^{3t}$.

Equation (4.1) is in fact derivable from a variational principle and one may obtain conservation laws via Noether's theorem using, e.g., the Lagrangian (Ibragimov *et al.*, 1998, for a symmetry method for the construction of Lagrangians)

$$L = \tfrac{1}{2}\, e^{2t}(u_tu_x + u^2u_t)$$

The Noether point symmetry generators associated with this L are deduced by invoking (2.8) with $B^i = B^i\,(t, x, u)$, $i = 1, 2$. One obtains three Noether symmetry generators (or a linear combination thereof)

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = e^t\frac{\partial}{\partial t} - ue^t\frac{\partial}{\partial u} - (ue^t + 2u_te^t)\frac{\partial}{\partial u_t} - u_xe^t\frac{\partial}{\partial u_x}$$

$$X_3 = \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - u_t\frac{\partial}{\partial u_t} - 2u_x\frac{\partial}{\partial u_x}$$

and these constitute a subalgebra of the four-dimensional Lie algebra of point symmetries of equation (4.1). The components B^1 and B^2 corresponding to the X_i, $i = 1,2,3$, are given by

$$B^1 = -\tfrac{1}{6}\,u^3e^{3t}\delta_{i2} + b(t,x), \qquad B^2 = -\tfrac{1}{4}\,u^2e^{3t}\,\delta_{i2} + c(t, x), \qquad i = 1, 2, 3$$

where b and c satisfy $b_t + c_x = 0$. The Noether theorem gives rise to conserved vectors for each of the Noether symmetry generators. They are, upon invoking (2.9) (we have set $b = c = 0$),

$$T^1 = -\tfrac{1}{2}u_x^2e^{2t} - D_x(\tfrac{1}{6}u^3e^{2t}), \qquad T^2 = -\tfrac{1}{3}u^3e^{2t} + D_t(\tfrac{1}{6}u^3e^{2t})$$

$$T^1 = -\tfrac{1}{3}u^3e^{3t} - D_x(\tfrac{1}{4}u^2e^{3t}), \qquad T^2 = -\frac{1}{2}e^{3t}(u_t + u)^2 + D_t(\frac{1}{4}u^2e^{3t})$$

$$T^1 = (u + xu_x)[\tfrac{1}{2}u_xe^{2t} + \tfrac{1}{2}u^2e^{2t}], \qquad T^2 = -\tfrac{1}{2}xu^2u_te^{2t} + \tfrac{1}{2}u_tue^{2t} + \tfrac{1}{2}u_t^2e^{2t}$$

The first two pairs yield conservation laws which are equivalent to what we obtained before. Note that there is no scaling symmetry associated with this Lagrangian. However, we have seen above that a scaling symmetry does not give rise to a conservation law. There is no alternative Lagrangian for which the scaling symmetry is a Noether symmetry. The Noether symmetry generator X_3 is a combination of scaling and a time translation and one obtains a corresponding (Noether) conserved vector given by the third pair above. However, the time-translation symmetry does not appear here as a Noether symmetry for the L considered. The question is whether it results in a conservation law. The answer can be provided by either looking for a Lagrangian for which $\partial/\partial t$ is a Noether symmetry generator (and hence a conservation law results) or alternatively by invoking (3.3) and (2.4) using $\partial/\partial t$. The last-mentioned possibility gives a negative answer to this question. The calculations are straightforward as before. Thus there is no first-order conservation law associated with the symmetry generator $\partial/\partial t$.

In the next example, which does not have a Lagrangian, we use the direct method to obtain a conservation law. Such a direct approach does not give the relationship between symmetries and conservation laws. Thus we need to invoke our symmetry conditions to determine the conservation laws that have associated symmetries.

Example 2. We first employ the direct method to derive a conservation law for Burger's equation

$$u_t = u_{xx} + uu_x \tag{4.5}$$

Thereafter we associate symmetries to the conservation law. The determining equation for the components T^1 and T^2, using the direct approach, is $(D_tT^1 + D_xT^2)|_{(4.5)} = 0$. The replacement of u_t by $u_{xx} + uu_x$ and then expansion and separation by u_{tt}, u_{tx}, and u_{xx} gives

$$\frac{\partial T^1}{\partial u_t} = 0\,, \qquad \frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial u_t} = 0\,, \qquad \frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial u_x} = 0 \tag{4.6}$$

together with the remaining terms

$$\frac{\partial T^1}{\partial t} + uu_x\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} = 0$$

The solution of the above linear equations is straightforward and yields

$$\begin{aligned} T^1 &= -\alpha(t)u_x + Au + \beta(t, x), \\ T^2 &= \alpha(t)u_t - Au_x + \dot{\alpha}u - \tfrac{1}{2}Au^2 + \gamma(t, x) \end{aligned} \tag{4.7}$$

where A is a constant and $\beta_t + \gamma_x = 0$. This conservation law is equivalent to the obvious one with conserved vector $(u, \frac{1}{2}u^2 + u_x)$.

Equation (4.5) has five point symmetries (in extended form):

$$X_1 = \frac{\partial}{\partial t}\,, \qquad X_2 = \frac{\partial}{\partial x}\,, \qquad X_3 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u} - u_x\frac{\partial}{\partial u_t}$$

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 3u_t\frac{\partial}{\partial u_t} - 2u_x\frac{\partial}{\partial u_x}$$

$$X_5 = tx\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - (tu + x)\frac{\partial}{\partial u} - (u + 3tu_t + xu_x)\frac{\partial}{\partial u_t} - (1 + 2tu_x)\frac{\partial}{\partial u_x}$$

We now invoke the symmetry condition (3.3) for each of the symmetries above on the conserved components (4.7).

It is easy to deduce that $X_1(T^1) = X_1(T^2) = 0$ imposed on (4.7) gives $T^1 = -\alpha u_x + Au + \beta(x)$, $T^2 = \alpha u_t - Au_x - \frac{1}{2}Au^2$. Without loss of generality, we choose

$$T^1 = u, \qquad T^2 = -u_x - \tfrac{1}{2}u^2 \tag{4.8}$$

which has associated symmetry X_1. Similarly for X_2, one gets (4.8) up to trivial terms.

The Galilean symmetry generated by X_3 together with the symmetry conditions on T^1 and T^2 of (4.7), viz.

$$X_3(T^1) = 0, \qquad X_3(T^2) - T^1 = 0$$

results in (up to inconsequential trivial terms)

$$T^1 = u + D_x\left(\frac{x^2}{2t}\right), \qquad T^2 = -u_x - \frac{1}{2}u^2 - D_t\left(\frac{x^2}{2t}\right) \tag{4.9}$$

Note that one still has terms in parentheses that give rise to a trivial conservation law. These terms play an important role in the Galilean symmetry associated with the conservation law (4.9). If T^1 and T^2 lack these terms, then X_3 will no longer be associated with T^1 and T^2.

The scaling symmetry generated by X_4 and the symmetry condition on (4.7)

$$X_4(T^1) + T^1 = 0, \qquad X_4(T^2) + 2T^2 = 0$$

give rise to

$$T^1 = u + (2/x)\,h(x^2/t), \qquad T^2 = -u_x - \tfrac{1}{2}u^2 + (1/t)\,h(x^2/t)$$

where h is a function of x^2/t. Again the term containing h is the trivial term without which one cannot associate X_4 with the conservation law of (4.7).

There is no conservation law corresponding to (4.7) which has associated with it the symmetry generated by X_5. To see this, one proceeds by solving

$$X_5(T^1) + tT^1 = 0, \qquad X_5(T^2) + 2tT^2 - xT^1 = 0$$

for T^1 and T^2 given by (4.7).

In the following example, we employ our symmetry-based approach to construct conservation laws for an equation that does not admit of a Lagrangian formulation. Here we invoke the dual conditions (2.4) and (3.3) simultaneously for one symmetry.

Example 3. One of the four Lie point symmetry generators admitted by the Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x \tag{4.10}$$

is the Galilean operator with first prolongation

$$X = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u} - u_x\frac{\partial}{\partial u_t}$$

It is worth noting that the Lie–Bäcklund algebra admitted by (4.10) is generated by X (Ibragimov, 1985). The determining equation for the conservation law of (4.10) after replacing u_{xxx} by $u_t - uu_x$ implies

$$\frac{\partial T^1}{\partial u_{tt}} = 0, \qquad \frac{\partial T^1}{\partial u_{tx}} + \frac{\partial T^2}{\partial u_{tt}} = 0, \qquad \frac{\partial T^1}{\partial u_{xx}} + \frac{\partial T^2}{\partial u_{tx}} = 0 \tag{4.11}$$

together with the remaining terms. Also, the symmetry condition (3.3) on T^1 and T^2 is

$$X(T^1) = 0, \qquad X(T^2) - T^1 = 0$$

The solution of these taking into account (4.11) gives

$$T^1 = T^1\,(a, b, c, \alpha, \beta), \qquad T^2 = -\gamma\frac{\partial T^1}{\partial c} - uT^1 + h(a, b, c, \alpha, \beta) \tag{4.12}$$

where $a = t$, $b = u_x$, $c = u_{xx}$, $\alpha = x + tu$, $\beta = u_t - uu_x$, and $\gamma = u_{tx} - 2uu_{xx}$. The remaining terms in the determining equation become

$$\begin{aligned}
&\frac{\partial T^1}{\partial a} + a\beta\frac{\partial T^1}{\partial \alpha} + [u_{tt} - u_tu_x - 2uu_{tx} + uu_x^2 + u^2u_{xx}]\frac{\partial T^1}{\partial \beta} + (\gamma + uu_{xx})\frac{\partial T^1}{\partial b}\\
&\quad -\gamma(1 + ab)\frac{\partial^2 T^1}{\partial c\partial\alpha} + (1 + ab)\frac{\partial h}{\partial \alpha} + \gamma(u_x^2 - u_{tx} + uu_{xx})\frac{\partial^2 T^1}{\partial c\partial\beta}\\
&\quad + (2bc + u\beta)\frac{\partial T^1}{\partial c} + (\gamma - b^2 + uu_{xx})\frac{\partial h}{\partial \beta} - bT^1 - \gamma c\frac{\partial^2 T^1}{\partial b\partial c}\\
&\quad + c\frac{\partial h}{\partial b} - \beta\gamma\frac{\partial^2 T^1}{\partial c^2} + \beta\frac{\partial h}{\partial c} = 0
\end{aligned}$$

After some straightforward manipulations, the solution of this equation gives

$$\begin{aligned}
T^1 &= (p(a, \alpha)a + D_1\alpha^2 + D_2\alpha + D_3)b + p\\
T^2 &= -(pa + D_1\alpha^2 + D_2\alpha + D_3)\beta + 2D_1c\alpha - 2bD_1\\
&\quad -ab^2D_1 + cD_2 + n\,(a, \alpha) - uT^1
\end{aligned} \tag{4.13}$$

where p and n satisfy $\partial p/\partial a + \partial n/\partial\alpha = 0$ and the D_i are constants. We discuss two nonequivalent cases of (4.13).

If $p = \alpha$, $n = 0$, $D_1 = D_3 = 0$, and $D_2 = 1$, then the components of the conservation law for (4.10) are

$$T^1 = (x + tu)(tu_x + u_x + 1), \qquad T^2 = u_{xx} - (x + tu)(tu_t + u + u_t)$$

This gives rise to a conservation law which is equivalent to

$$D_t(xu_x + tuu_x) + D_x(-xu_t - tuu_t + u_{xx}) = 0$$

which in turn can be obtained by setting $D_2 = 1$ and the rest of the terms zero.

Next we assume that $D_2 = D_3 = 0$, $D_1 = -1/2$, and $p = \alpha^2/(2a)$. Then

$$T^1 = xu + \frac{tu^2}{2} + \frac{x^2}{2t} \tag{4.14}$$

$$T^2 = u_x + \frac{1}{2}tu_x^2 - xu_{xx} - tuu_{xx} - \frac{1}{2}xu^2 - \frac{1}{3}tu^3 + \frac{1}{6}\frac{x^3}{t^2}$$

This is precisely (up to trivial terms) the components of the conservation law given in Ibragimov (1985). In Ibragimov (1985), (4.14) was obtained from a *weak* Lagrangian formulation. The components (4.14) and the recursion operator of (4.10), viz., $L = D^2 + \frac{2}{3}u + \frac{1}{3}u_x D^{-1}$, enable one to determine all the conserved vectors of (4.10) (see discussion in Ibragimov, 1985).

In the next example we illustrate how one can find the point symmetries associated with a conservation law.

Example 4. We determine the point symmetries associated with the conserved components

$$T^1 = xu_x + tuu_x, \qquad T^2 = u_x - xu_t - tuu_t \tag{4.15}$$

of the Burgers equation. Clearly this gives rise to a conservation law equivalent to $D_t(-u) + D_x(u^2/2 + u_x) = 0$ since $T^1 = -u + D_x(xu + \frac{1}{2}tu^2)$ and $T^2 = -u^2/2 + u_x - D_t(xu + \frac{1}{2}tu^2)$.

The symmetry conditions on (4.15) are

$$X(T^1) + T^1 D_x(\xi^2) - T^2 D_x(\xi^1) = 0, \qquad X(T^2) + T^2 D_t(\xi^1) - T^1 D_t(\xi^2) = 0 \tag{4.16}$$

where X is the operator (2.5), viz.

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x}$$

The expansion of the determining equations (4.16) and separation by monomials of the first derivatives gives

$$\xi^1 = -2ta, \qquad \xi^2 = -bt - ax, \qquad \eta = au + b$$

where a and b are constants. Hence there are two symmetries associated with T^1 and T^2 and their generators are

$$X_1 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u} - u_x\frac{\partial}{\partial u_t},$$

$$X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} - 3u_t\frac{\partial}{\partial u_t} - 2u_x\frac{\partial}{\partial u_x}$$

Note that the symmetries associated with the conservation law given by (4.15) and that with the equivalent one determined by $(-u, u^2/2 + u_x)$ are quite different. The trivial conservation law plays an important role in the association of symmetries to conservation laws.

5. DISCUSSION

The relations (3.3) and its consequences have important applications. First, (3.3) could be utilized to obtain the symmetries X associated with a given conserved vector T as illustrated in Section 4. In the case of ordinary differential equations, this has been of interest (Kara *et al.*, 1994) and provides reduction procedures. Second, (3.3) with X known together with the conservation law $D_i T^i = 0$ can be viewed as a system of linear partial differential equations which can be solved for the components T^i of the conserved vector T. This aspect was investigated in Section 4 for some well-known partial differential equations. However, in the literature the problem of constructing a conserved vector is generally obtained by means of the *direct method* of solving $D_i T^i = 0$ for a given equation without recourse to symmetry properties and usually involves ad hoc assumptions to simplify the solution procedure. The conditions (3.3) added to $D_i T^i = 0$, satisfied on the solutions of the system under investigation, imposes natural symmetry conditions and the resulting determination of the conserved vector $T = (T^1, \ldots, T^n)$ becomes simpler. Also the conditions (3.3) provide a relationship between symmetries and conservation laws. Finally, there have been applications of special cases of (3.3) involving the variational formulation of a differential equation in the construction of a corresponding Lagrangian (Ibragimov *et al.*, 1998).

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